Scaling Quantum Networks: Inter-QLANs Artificial Connectivity

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***Abstract*—The Quantum Internet is envisioned, at its final stage, to globally interconnect heterogeneous quantum networks. And this interconnection relays on entangled states shared among several quantum networks. In this paper, we take a step towards the engineering of entanglement as a inter-network resource. Specifically, we consider the interconnection of Quantum Local Area Networks (QLANs), and we show how to dynamically generate inter-QLAN *artificial* connectivity by means of local operations only. To this aim, we first design the multipartite entangled state to be distributed within each QLAN. Then, we show how such a state can be engineered for obtaining *artificial links* between different QLANs nodes. By leveraging the properties of a particular class of multipartite entangled states, namely, *graph states*, our results dynamically adapt the artificial connectivity to the network traffic patterns. Our proposal is preliminary towards scaling the interconnection of quantum networks.**

***Index Terms*—Entanglement, Quantum Networks, Quantum Communications, Quantum Internet**

I. Introduction

There is a common agreement on entanglement being the key resource for the Quantum Internet [1]–[4]. Indeed, entanglement enables a new form of connectivity among remote nodes, even if they are not connected through a direct physical quantum channel. For this reason, entangled states – such as EPR pairs – enable *artificial communication links* among the quantum network nodes. More into details, an artificial link represents a virtual communication link – established between two remote nodes that share some entanglement – which can be exploited to perform communication tasks between the two remote nodes.

In this context, entanglement distribution plays a crucial role. Indeed, several strategies can be adopted. As instance, one could distribute an EPR pair between any couple of nodes, which results in an all-to-all connectivity. However, this strategy is unfeasible for large networks, since it does not scale. Alternatively, entanglement could be distributed according to a strategy reminiscent of *reactive* classical network routing. Specifically, EPR pairs could be distributed along the path connecting the source and the destination, whenever a communication request occurs. Then, with swapping operations at the intermediate nodes, an artificial link between the source and the destination is eventually built, by consuming EPR pairs along the path. Unfortunately, this strategy presents majors drawbacks such as severe delays and network overhead. Indeed, before serving a communication request, several operations must be successfully concluded including – but not limited to – entanglement generation, distribution and swapping. Furthermore from a network engineering perspective, it is important to highlight that the identities of the nodes to be artificially linked must be decided a-priori.

Thankfully, multipartite entanglement represents a powerful resource allowing to decide *on-demand*, i.e. at run-time, the identities of the nodes exploiting the communication resource. For this reason, the connectivity enabled by such states is referred to as *on-demand connectivity* [5]. To elaborate more, a multipartite entangled state shared among a set of remote nodes allows to dynamically extract one – or more – EPR pair by only exploiting Local Operations and Classical Communications (LOCC). This allows to dynamically select the identities of the nodes exploiting the communication resource, without any additional delay [4], [5].

Within the study of multipartite entangled states, some notable works have been proposed. These concern the analysis of their quantum properties, such as their amount of entanglement [6] and their so called *pairability*, namely, the ability of extracting EPRs from a multipartite state [7]–[9]. Other notable works focused on sufficient conditions for a multipartite entangled state to be defined as k*-universal*, which represent a generalization of the aforementioned pairability measurement [10]. Besides, in [11] the authors propose a class of multipartite entangled state, namely, graph states, for the implementation of an all-photonic quantum repeater for quantum networks.

Despite the aforementioned remarkable works, multipartite entangled states have been poorly investigated so far from a network engineering perspective. On the contrary, in this paper, we take a step towards the engineering of multipartite entanglement as a communication resource for quantum networks. To this aim, we consider the interconnection of Quantum Local Area Networks (QLANs) and we engineer a multipartite entangled state for inter-QLANs connectivity. Then, stemming from the designed multipartite state, we show how to build upon the physical topologies, through local operation only, several *artificial topologies* ranging from the fully connected topology to the star topology, each involving different-QLANs nodes.

Hence, stemming on these results, we proactively adapt

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Angela Sara Cacciapuoti acknowledges PNRR MUR NQSTI-PE00000023, Marcello Caleffi acknowledges PNRR MUR project RESTART-PE00000001.

979-8-3315-4137-8/24/$31.00 ©2024 IEEE
DOI 10.1109/QCE60285.2024.00229

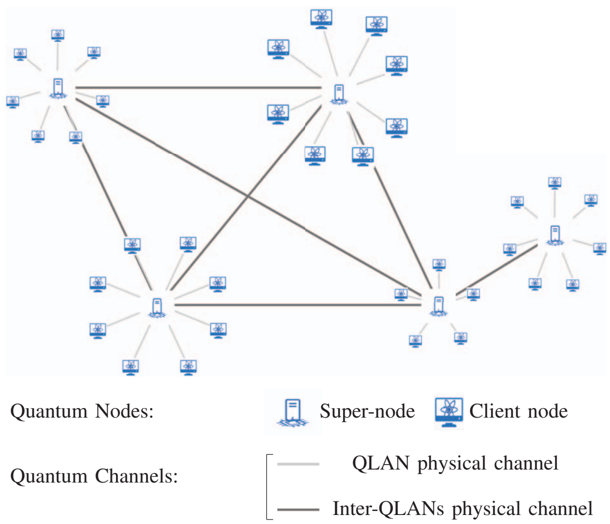


Fig. 1: Schematic representation of the considered quantum network architecture. The network comprises several Quantum Local Area Networks (QLANS). Within each QLAN, a super-node generates and distributes resources – namely, multipartite entangled states – to a set of quantum nodes – referred to as *clients* – with a star-like topology. Inter-QLAN connectivity is enabled by point-to-point quantum channels interconnecting different super-nodes.

the artificial connectivity among different QLANs so that the networks can easily adapt to different traffic patterns or requirements. Notably, this represents a crucial ability required for an effective quantum network design.

II. Preliminaries

A. Graph States

Graph states can be effectively described with graph theory tools. Specifically, stemming from an arbitrary graph $G = (V, E)$, the corresponding *graph state* – denoted with $|G\rangle$ – can be obtained by mapping each vertex of the graph G with a qubit in the state $|+\rangle$, and by performing a controlled-Z (CZ) gate between each pair of qubits corresponding to adjacent vertices in G [12], [13]. Since the CZ operation is an entangling operation, an edge corresponds to quantum correlation shared between two qubits. As a consequence, the distribution of a graph state among remote nodes of a quantum network establishes an entanglement-based connectivity between remote nodes.

Formally, the graph state $|G\rangle$ associated to graph $G = (V, E)$ is obtained as[1]:

$$|G\rangle = \prod_{(i,j)\in E} \mathtt{CZ}_{(i,j)} |+\rangle^{\otimes n} \tag{1}$$

where $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ denotes the initial qubit state, $n = |V|$ and $CZ_{(i,j)}$ denotes the CZ gate applied to the qubits associated to the neighbour vertices $i, j \in V$.

Remarkably, although a graph state uniquely correspond to a graph, there exist some equivalence classes among such states. From a network engineering perspective, an equivalency class of interest is represented by the so-called *local unitary* (LU) equivalence, which constitutes a key metric for quantifying and classifying entanglement [14]. As instance, the n-qubit graph state $|S_n\rangle$ corresponding to the star graph S_n (formally defined in Def. 3) and the n-qubit GHZ state [15], [16] are LU-equivalent.

The mapping between graph states and graphs is of a paramount importance, beyond the expression in (1). Specifically, the action of fundamental operations – such as Pauli measurements – on a graph state $|G\rangle$ can be described via simple transformations on the associated graph G.

More into details, a projective measurement through one of the Pauli operators – namely, σ_x, σ_y, or σ_z – on a qubit of the graph state $|G\rangle$ yields to a new graph state $\left|\tilde{G}\right\rangle$ on the unmeasured qubits. As proved in [6], [12], this new graph state $\left|\tilde{G}\right\rangle$ can be obtained – up to local unitaries – by means of simple transformations on the graph G associated to the original graph state $|G\rangle$, such as vertex deletion and the local complementation. Since projective measurements will be exploited in Sec. III for engineering the artificial connectivity enabled by entanglement, it is convenient to summarize in the following their effects on an arbitrary graph state $|G\rangle$ [6], [12].

Pauli Measurements. *The projective measurement via a Pauli operator σ_ξ^i on the i-th qubit of the graph state $|G\rangle$ – namely, on the qubit associated to vertex i in graph G – yields to a new graph state $\left|\tilde{G}\right\rangle$ among the remaining qubits, which is LU-equivalent to the graph state $|G'\rangle$, associated to graph G' obtained with vertex deletion and local complementations:*

$$G' \equiv \begin{cases} G - i & \text{if } \xi = z \\ \tau_i(G) - i & \text{if } \xi = y \\ \tau_{k_0}\big(\tau_i\big(\tau_{k_0}(G)\big) - i\big) & \text{if } \xi = x. \end{cases} \tag{2}$$

In (2), $G - i$ denotes vertex i deletion and $\tau_i(\cdot)$ denotes the local complementation[2] operation at vertex i, and $k_0 \in N_i$ denotes an arbitrary neighbor of vertex i.

III. System model

A. Network Topology

We consider, as the archetype of the future Quantum Internet, the network resulting from the interconnection of different Quantum Local Area Networks (QLANs).

Entanglement generation is a complex hardware-demanding task, that becomes even more challenging when it comes to multipartite entanglement. For this, as commonly adopted in literature [17]–[21], it is pragmatic to assume each QLAN

[1] With a (widely-used) notation abuse in (1), since the application of the $\mathtt{CZ}_{(i,j)}$ gate on the state $|+\rangle^{\otimes n}$ requires a reference to $n-2$ identity operations I acting on all the qubits different from i or j.

[2] The local complementation at vertex i consists in complementing the edges among the nodes adjacent to i. In a nutshell, it requires to remove all the edges that were previously connecting such nodes, and to add edges that were not previously connecting such nodes. For more details please refer to [13].

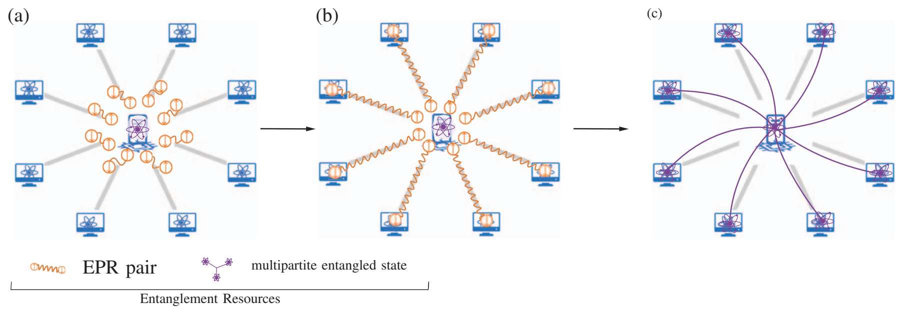


Fig. 2: Pictorial representation of the multipartite entanglement distribution process within a single QLAN of Fig. 1. (a) The super-node is responsible for entanglement generation and distribution within each QLAN. Accordingly, it locally generates the multipartite entanglement state and distribute it via teleportation. For this, one additional EPR pair per client must be generated at the super-node. (b) Once an EPR pair is shared between super-node and each client, one e-bit of the multipartite entangled state can be teleported to the client by consuming such an EPR. (c) Eventually, the multipartite entangled state is distributed to the clients so that all the QLAN nodes, including the super-node, are entangled.

organized in a star-like topology, as represented in Fig. 1, with a set of *clients* connected to a specialized *super-node*, which is responsible for entanglement generation and distribution.

Accordingly, multipartite entangled states are generated locally at each super-node, and then distributed to the corresponding clients via teleportation process, as represented in Fig. 2. The rationale for this strategy – namely, for distributing multipartite entangled states via teleportation rather than via direct transmission – lies in the higher robustness against losses and in the higher tolerance to different persistence levels exhibited by different classes of multipartite entangled states [21]–[23].

While intra-QLAN topologies are pragmatically assumed as star-like topologies for the reasons above, no constrains are enforced to inter-QLAN connectivity, which is enabled by point-to-point quantum links interconnecting different super-nodes as shown in Fig. 1.

B. Problem Statement

Stemming from the network architecture introduced so far, we can now formally define our problem, by focusing on the toy-model constituted by two QLANs interconnected by a single link between the corresponding super-nodes.

Problem. *Given two QLANs, interconnected by a single physical link between the corresponding super-nodes, the goal is to design and engineer a multipartite entanglement state distributed in each QLAN so that artificial links among nodes belonging to different QLANs can be dynamically obtained on-demand, by overcoming so the constraints induced by the physical topology.*

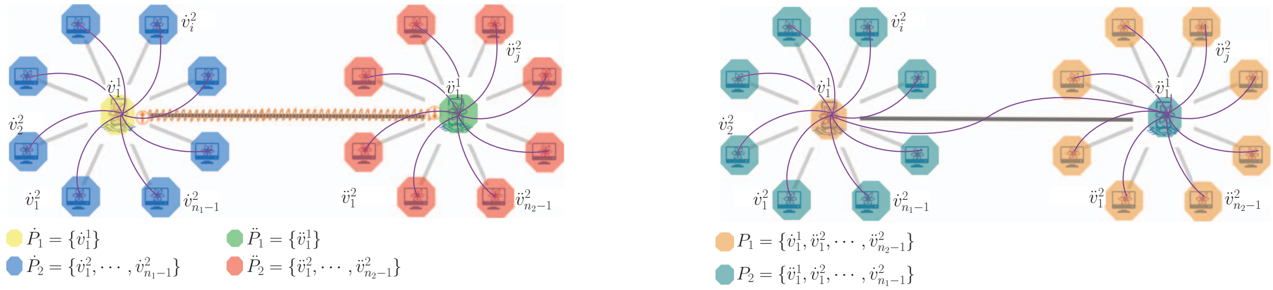
In essence, an artificial link represents a virtual communication link established between two remote nodes, since they share some entanglement.

In EPR-based networks, artificial links between distant nodes can be obtained by relying only on bipartite entanglement and thus by performing swapping operations at intermediate nodes [4] so that an EPR pair between remote nodes is eventually obtained. Yet, this strategy presents a drawback: the identities of the nodes to be artificially linked must be decided a-priori. In other words, for each EPR pair[3] distributed through the inter-QLAN link only one artificial link among distant nodes can be obtained. This implies that artificial links via entanglement swapping is reminiscent of *reactive* classical routing strategies, where the source-destination path is discovered when a packet is ready to be transmitted.

Conversely, in multipartite-based networks, multiple artificial links between distant nodes can be obtained by properly choosing the initial multipartite state and by wisely manipulating it via local operations – i.e., via *free operations* from a quantum communication perspective. In such a way, we can pro-actively generate and distribute entanglement among subset of nodes of different QLANs so that the identities of the nodes eventually communicating can be chosen dynamically at run time. Clearly, this strategy is reminiscent of *proactive* classical routing strategies, where source-destination paths are discovered in advance, and they remain ready to be used eventually, when the necessity of transmitting a packet arises.

As an example, let us consider two adjacent QLANs in Fig. 1, by assuming – for the sake of exemplification – a traffic pattern from super-node of the leftmost QLAN to clients of the rightmost. An EPR shared between super-node and any of the clients can be obviously extracted via entanglement swapping. Yet, the identity of the client must be decided a-priori. Conversely, we aim at generating an artificial connectivity as the one represented in Fig. 4a, where artificial links between super-node and each remote client are proactively generated. This artificial connectivity will be eventually manipulated – without the need of further quantum communications – when a communication request will be ready to be served by extracting the ultimate artificial link interconnecting the effective source-destination pair.

[3]Hereafter, we obviously refer to maximally-entangled EPR pairs, neglecting any noise affecting the EPR generation and distributing for the sake of exposition simplicity.



(a) Each super-node generates and distributes, as described in Fig. 2, a star graph state in each QLAN, denoted as $\left|\dot{S}_{n_1}\right\rangle = (\dot{P}_1, \dot{P}_2, \dot{E})$ and $\left|\ddot{S}_{n_1}\right\rangle = (\ddot{P}_1, \ddot{P}_2, \ddot{E})$, respectively. Furthermore, an EPR is generated and shared between the two super-nodes.

(b) By exploiting the EPR shared between the two QLANS for performing a remote CZ, a new graph state shared among all the nodes of the two QLANs is obtained. Remarkably, also this new graph state is a two-colorable graph state, and specifically it is a binary star graph $|S_{n_1,n_2}\rangle$.

Fig. 3: Interconnecting two QLANs through a binary graph state $|S_{n_1,n_2}\rangle$ distributed among all the nodes, obtained from two star graphs distributed in each QLAN and an additional EPR pair.

IV. Inter-QLANs Artificial Connectivity

A. Engineering Multipartite Entanglement

We aim at engineering the entanglement-based artificial topology for creating artificial links among network remote nodes belonging to different QLANs. To this aim, the choice of the initial multipartite entangled state distributed in each QLAN is of paramount importance.

As discussed in Sec. II-A, we focus our attention on graph states due to the useful mapping between operations on a graph state $|G\rangle$ and transformations of the associated graph G. Yet, graph states represents a wide class of multipartite entangled states. In the following, we design and manipulate a specific instance of graph states that allows us to addresses our problem: dynamically enabling multiple artificial links among distant nodes. Before formally introducing this specific graph state in Def. 4, the following preliminaries are needed.

Definition 1 (Two-colorable Graph or Bipartite Graph). A graph $G = (V, E)$ is two-colorable if the set of vertices V can be partitioned[4] into two subsets $\{P_1, P_2\}$ so that there exist no edge in E between two vertices belonging to the same subset. Two-colorable graph $G = (V, E)$ can be also denotes as $G = (P_1, P_2, E)$.

We focus our attention on two-colorable graph states without loss of generality, since any graph state can be converted in a two-colorable one under relaxed conditions [6]. Indeed, any graph is two-colorable iff it does not contain cycles of odd length. Furthermore, two-colorable graphs model a wide range of different network topologies, including bus, ring and star [13]. In the following, for the sake of notation simplicity, we label the vertices in P_1 and P_2 as follows:

$$P_1 = \{v_1^1, \cdots, v_{n_1}^1\} \wedge P_2 = \{v_1^2, \cdots, v_{n_2}^2\} \tag{3}$$

with $n_1 + n_2 = n$.

[4]A partition of a set is a grouping of its elements into non-empty subsets, in such a way that every element is included in exactly one subset.

Definition 2 (Complete Bipartite Graph). Let $G = (P_1, P_2, E)$ be a bipartite graph with $|P_1| = n_1$ and $|P_2| = n_2$. If $E = P_1 \times P_2$, i.e., if

$$\forall\, v_i^1 \in P_1 \wedge v_j^2 \in P_2, \exists\, (v_i^1, v_j^2) \in E, \tag{4}$$

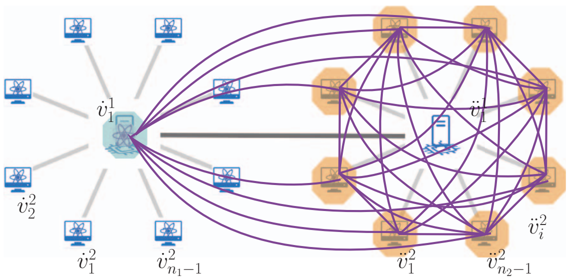
G is defined as *complete bipartite graph* and denoted as K_{n_1,n_2}.

Hence, in a complete bipartite graph, any vertex belonging to one part is connected to every vertex belonging to the complementary part by one edge.

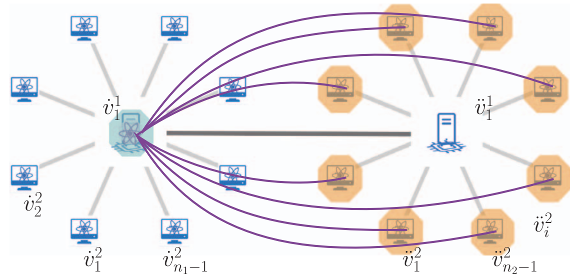
Definition 3 (Star Graph). Let K_{n_1,n_2} be a complete bipartite graph. If n_1 (or equivalently n_2) is equal to 1, then the graph is called *star graph* and denoted equivalently as either $K_{1,n-1}$ or S_n, where $n-1$ is the cardinality of the other part. In the following, we define vertex v_1^1 as the *center* of the star graph.

QLAN Entanglement Resource. *The star graph state $|S_n\rangle$ represents the multipartite entangled state generated and distributed in each QLAN, with each qubit of state $|S_n\rangle$ distributed to a different node. Specifically, by following labelling* (3)*, qubit corresponding to vertex v_1^1 is stored at the super-node, whereas qubits corresponding to vertices $\{v_1^2, \ldots, v_{n-1}^2\}$ are distributed to the clients.*

Such a state corresponds to a graph that perfectly matches with the QLAN physical topology and it is easy to generate [18], [20], [24]. It is worthwhile to mention that it represents the worst-case scenario, since from a star graph it is possible to extract only one EPR pair, thus limiting the communication dynamics within the single QLAN. Despite this, in the next sections we will prove that by properly manipulating the multipartite states in the different QLANs, the limitations of the physical topologies can be overcome. Stemming from the concept of star graph S_{n-1}, we are ready now to introduce a two-colorable graph state that will be extensively used in the following, namely, the *binary star graph state* $|S_{n_1,n_2}\rangle$.



(a) *Hierarchical peer-to-peer* artificial topology discussed in Prop. 2: an artificial fully-connected topology among all the clients of the same QLAN and one super-node of a different QLAN.



(b) *Clients hand-over* artificial topology discussed in Cor. 1: an artificial star topology among the same set of nodes of Fig. 4a, but centered at a super-node of a different QLAN.

Fig. 4: Different artificial inter-QLAN topologies matching with a traffic pattern involving the super-node of one QLAN and clients of the other QLAN. Remarkably, all the artificial inter-QLAN topologies are obtained by manipulating a binary star graph state with local operations and measurements only.

Definition 4 (Binary Star Graph). A binary star graph S_{n_1,n_2} is a bipartite graph $G = (P_1, P_2, E)$ with the edge set E defined as:

$$E = \{v_1^1\} \times P_2 \cup P_1 \times \{v_1^2\} \tag{5}$$

with P_1 and P_2 given in (3).

From (5), it results that only one vertex in each part of a binary star graph – $v_1^1 \in P_1$ and $v_1^2 \in P_2$ – is fully connected.

In Sec. IV-B, we will show that – by locally manipulating at some specific network nodes a binary star state $|S_{n_1,n_2}\rangle$ shared between the two QLANs, artificial links among remote nodes are dynamically generated. Yet, before operating on such a state, the state must be distributed among all the nodes. Hence, one preliminary question naturally arises: *how expensive is it – from a quantum communication perspective – to distribute such a state within the two QLANs*? Or, in other words, how many EPR pairs must be consumed for distributing such a state? One might believe that the required number of EPR pairs should somehow depend on the number of artificial links that must be generated among remote nodes belonging to different QLANs.

We answer to this question with the following proposition.

Proposition 1. *Let's assume that a star state* $\left|\dot{S}_{n_1}\right\rangle$ *has been distributed in the first QLAN and that another star state* $\left|\ddot{S}_{n_2}\right\rangle$ *has been distributed in the second QLAN. Then, a binary star state* $|S_{n_1,n_2}\rangle$ *can be distributed among all the nodes by consuming only one EPR pair at the two super-nodes.*

Proof: Please refer to App. A ■

In Fig. 3 we represents the binary star state building process, by also labelling each network node with the (vertex corresponding to the) stored qubit. Thus, we can now to define our global entanglement resource.

Inter-QLAN Entanglement Resource. *The binary star state* $|S_{n_1,n_2}\rangle$ *represents the inter-QLAN multipartite entangled resource, which is locally manipulated at network nodes for dynamically enabling multiple artificial links among remote nodes.*

B. On-Demand Artificial Links

Here, we show that multiple artificial links can be dynamically obtained among remote nodes belonging to different QLANs, by means of local operations only. Specifically, the set of employed operations limits to single qubit gates, single qubit Pauli measurements and classical communications, which all represent *free operations* from a quantum communication perspective.

To this aim, we consider two different archetypes of traffic patterns, *super-node to clients* and *client to clients*. For each type, we discuss different artificial topologies that satisfy the communication demand.

Proposition 2 (Hierarchical Peer-to-Peer). *A binary star graph state* $|S_{n_1,n_2}\rangle$ *shared between* $n = n_1 + n_2$ *nodes belonging to two different QLANs allows to obtain a* n_i*-complete graph state* $|K_{n_i}\rangle$ *(with* $i = 1, 2$*) connecting the super-node of a QLAN and all the* $(n_i - 1)$*-clients of the other QLAN.*

Proof: Please refer to App. B ■

The result of Prop. 2 is particularly relevant from a communication perspective. Specifically, the fully connected graph represented in Fig. 4a corresponds to a GHZ state. As already acknowledged, GHZ state allow to extract one EPR pair between *any couple* of nodes sharing it [21]. As a consequence, they exhibit a remarkable flexibility on the choice of the identities of the nodes exploiting the communication link. Notably, the completely connected graph includes the other QLAN super-node. Hence, from a topological perspective, the involved clients and super-node act as peer-to-peer entities, which can exploit the shared entangled state either to accommodate intra-QLAN traffic requests or inter-QLAN traffic requests. This consideration induced us to label this proposition as "hierarchical peer-to-peer" artificial topology, by including the differentiation – aka hierarchy – in terms of hardware requirements between clients and super-node. The hierarchical peer-to-peer artificial topology is particularly advantageous whenever no information is available on the actual network traffic features of the QLAN clients. Specifically, if a client may equally need to communicate with clients belonging to the same QLANs or with clients belonging to a different QLAN, then the communication request will be ready to be served by proactively manipulating the "hierarchical peer-to-peer" artificial topology. And, remarkably, the communication request is easily served by simply performing local unitaries and measurements, without any additional quantum communication.

From Prop. 2, the following corollary follows.

Corollary 1 (**Client Hand-Over**). *A binary star graph state $|S_{n_1,n_2}\rangle$ shared between $n = n_1 + n_2$ nodes belonging to two different QLANs allows to obtain a n_i-nodes star graph state $|S_{n_i}\rangle$ (with $i = 1, 2$), centered at one QLAN super-node and connecting all the $(n_i - 1)$-clients of the other QLAN.*

Proof: The proof follows by adopting similar reasoning as in Prop. 2 and by replacing the Pauli-y measurement at the super-node with a Pauli-x measurement. ∎

We note that Cor. 1 creates artificial links between the super-node of one QLAN and the clients of the other QLAN. This, from a topological perspective, is equivalent to virtually *move* the clients of a QLAN into a different QLAN, resembling thus a sort of *client hand-over* from one QLAN to the other.

The artificial topologies represented in Fig. 4 are obtained by leveraging suitable sequences of Pauli measurements, which, thus, represent a tool for engineering the artificial connectivity. To elaborate more, the graph represented in Fig. 4a leverages a sequence of Pauli-z and Pauli-y measurements. Remarkably, by replacing the Pauli-y measurement at the super-node with a Pauli-x measurement, we obtain the LU-equivalent graph state corresponding to the clients hand-over topology, as shown in Fig. 4b.

The following Prop. 3 and the subsequent Cor. 2 prove that artificial topologies involving only clients belonging to different QLANs can be built by properly manipulating the binary star graph state.

Proposition 3 (**Pure Peer-to-Peer**). *A binary star graph state $|S_{n_1,n_2}\rangle$ shared between $n = n_1 + n_2$ nodes belonging to two different QLANs, allows to obtain a n_i-complete connected graph state $|K_{n_i}\rangle$ (with $i = 1, 2$) shared between one client node of a QLAN and all the $(n_i - 1)$-clients of the other QLAN.*

Proof: Please refer to App. C ∎

As showed in Fig. 5a, Prop. 3 allows an arbitrary client belonging to a QLAN to share an artificial link with any client belonging to a different QLAN. Hence, it generates an artificial QLAN topology among *peer* client entities – thus, the naming "pure" – by neighbouring remote nodes, despite the original constraints imposed by the physical topologies.

A pure peer-to-peer artificial topology extends the flexibility on the choice of the identities of the nodes exploiting the ultimate artificial link, by involving only clients at different QLANs. This could be particularly advantageous for designing distributed network functionalities relying on clients communication capabilities. Indeed, if a client needs to communicate with a client belonging to a different QLAN, then – by proactively manipulating the artificial topology – the communication request is ready to be served, without further orchestration at the super-node. Indeed, the communication request is fulfilled by performing local unitaries without any additional quantum communications. And, actually, the client of the other QLAN can be selected by properly manipulating the initial binary star graph. Hence their identities can be engineered on-demand.

Corollary 2 (**Role Delegation**). *Starting from a binary star graph state $|S_{n_1,n_2}\rangle$ shared between $n = n_1 + n_2$ nodes belonging to two different QLANs, a n_i-star graph state $|S_{n_i}\rangle$ (with $i = 1, 2$) centered at one QLAN client node and connecting all the remaining $(n_i - 2)$-clients of the same QLAN and a client node of the other QLAN can be obtained.*

Proof: The proof follows by adopting similar reasoning as in Prop. 3 and by replacing the Pauli-y measurement with a Pauli-x measurement. ∎

Due to the particular structure of this artificial topology shown in Fig. 5b – which has one client as center of the star graph instead of the super-node – we are induced to label this topology as "role delegation topology".

V. Discussion and Conclusions

We now provide the reader with an overall view of the derived result from a network perspective. As discussed, Pauli measurements – with the assistance of local unitaries and classical communications – represent an effective and powerful engineering tool able to radically change the network topology. Specifically, the action of measuring a qubit stored at an arbitrary nodes results in *disconnecting* such a node from the final virtual topology. This action is evident, as instance, in Fig. 4a, where Pauli-z measurements are performed at the clients of the leftmost QLAN. However, a different Pauli measurement – say x or y – doesn't limit to disconnect the measured node, but it changes the artificial topology by either creating additional edges among nodes previously disconnected or removing edges already present. And these changes are not limited to neighbor nodes, but they involve remote nodes as well. Accordingly, the effect of Pauli measurements allows to generate an entangled-based topology interconnecting clients and/or super-nodes belonging to different physical QLANS, without any further use of quantum links. This is equivalent to neighbour remote nodes, by exploiting the entanglement features [5]. In conclusion, by engineering the sequence and type of Pauli measurements to be performed, we exploit the potentialities of multipartite entanglement in "shaping" the artificial connectivity, as well as the freedom in the choice of the identities of the involved nodes.

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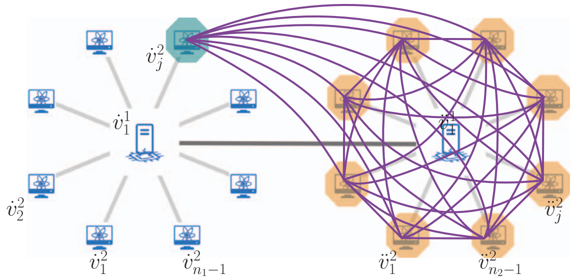
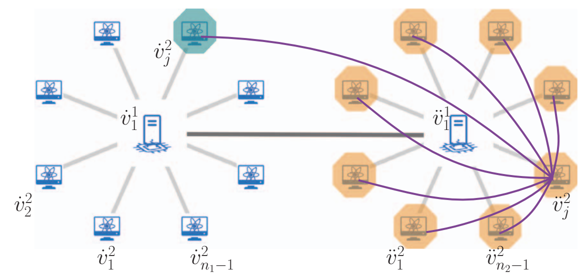


(a) *Pure peer-to-peer* artificial topology discussed in Prop 3: an artificial fully-connected topology among all the clients of the same QLAN and one client node of a different QLAN.



(b) *Role delegation* artificial topology discussed in Cor. 2: an artificial star topology among the same set of nodes of Fig. 5a, but centered at a client of the rightmost QLAN.

Fig. 5: Artificial inter-QLAN topologies matching with a traffic pattern involving one client of one QLAN and clients of the other QLAN. Remarkably, the action of "sacrificing" the artificial link between the two QLAN super-nodes enables inter-QLAN peer-to-peer artificial topologies particularly useful for designing distributed network functionalities relying on client communication capabilities.

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Appendix A
Proof of Proposition 1

Let us label the vertices of the graphs $\dot{S}_{n_1} = (\dot{P}_1, \dot{P}_2, \dot{E})$ and $\ddot{S}_{n_2} = (\ddot{P}_1, \ddot{P}_2, \ddot{E})$, associated to the graph states $\left|\dot{S}_{n_1}\right\rangle$ and $\left|\ddot{S}_{n_2}\right\rangle$ as:

$$\dot{P}_1 = \{\dot{v}_1^1\} \wedge \dot{P}_2 = \{\dot{v}_1^2, \cdots, \dot{v}_{n_1-1}^2\}, \tag{6}$$

$$\ddot{P}_1 = \{\ddot{v}_1^1\} \wedge \ddot{P}_2 = \{\ddot{v}_1^2, \cdots, \ddot{v}_{n_2-1}^2\}, \tag{7}$$

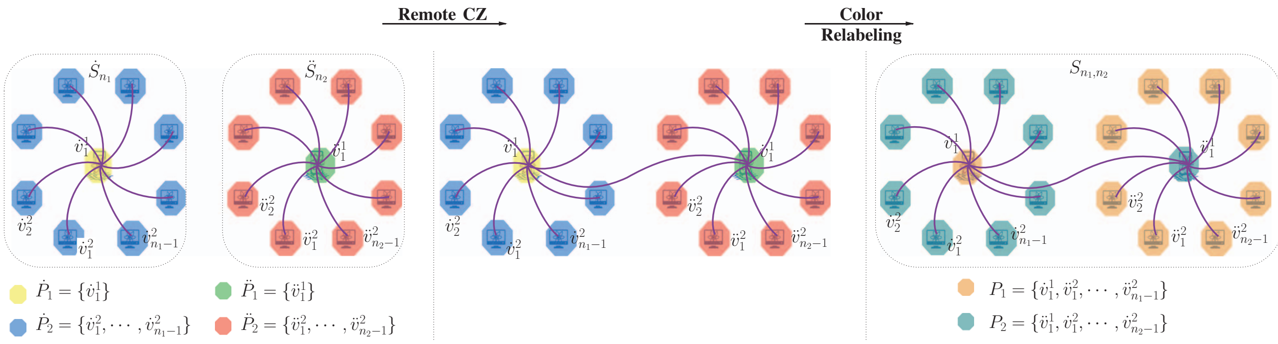
with $\dot{E} = \dot{P}_1 \times \dot{P}_2$ and $\ddot{E} = \ddot{P}_1 \times \ddot{P}_2$. Accordingly and as shown in Fig. 6-a, the qubits of graph state $\left|\dot{S}_{n_1}\right\rangle$ are distributed among the nodes of the first QLAN, with the super-node storing the qubit associated to vertex $\dot{v}_1^1$ and the clients storing the qubits associated to $\dot{v}_1^2, \ldots, \dot{v}_{n_1-1}^2$. Similarly, the qubits of the graph state $\left|\ddot{S}_{n_2}\right\rangle$ are distributed among the nodes of the second QLAN, with the super-node storing the qubit associated to the vertex $\ddot{v}_1^1$ and the clients storing the qubits associated to $\ddot{v}_1^2, \ldots, \ddot{v}_{n_1-1}^2$. By consuming an EPR pair, the two super-nodes can perform a `CZ` operation between the two qubits at their sides, which corresponds to adding edge $(\dot{v}_1^1, \ddot{v}_1^1)$, as shown in Fig. 6-b. Hence, this additional edge connects the (only) vertex in $\dot{P}_1$ with the (only) vertex in $\ddot{P}_2$:

$$E = \dot{E} \cup \ddot{E} \cup \{(\dot{v}_1^1, \ddot{v}_1^1)\}. \tag{8}$$

If we want to color the overall graph, then these two vertex sets must be colored with two different colors, say orange and cyan. However, no edges connect the vertex in $\dot{P}_1$ with vertices in $\ddot{P}_2$, and hence all these vertices can be colored with the same color, orange. Similarly, no edges connect vertex in $\ddot{P}_1$ with the vertices in $\dot{P}_2$, hence all these vertices can be colored with the same color. It follows that the overall graph G is a two-colorable graph with parts P_1 and P_2 given by:

$$P_1 = \dot{P}_1 \cup \ddot{P}_2 = \{\dot{v}_1^1, \ddot{v}_1^2, \cdots, \ddot{v}_{n_2-1}^2\} \tag{9}$$

$$P_2 = \ddot{P}_1 \cup \dot{P}_2 = \{\ddot{v}_1^1, \dot{v}_1^2, \cdots, \dot{v}_{n_1-1}^2\} \tag{10}$$



(a) (b) (c)

Fig. 6: Generation of a *binary star state* starting from two *star states* distributed in each QLAN, with physical topology (quantum links) omitted for the sake of simplicity. (a) Initial scenario where three entangled resources are shared: i) star state $\dot{S}_{n_1}$ among QLAN1 nodes, ii) star state $\ddot{S}_{n_2}$ among QLAN2 nodes, and iii) an EPR pair between the two super-nodes $\dot{v}_1^1$ and $\ddot{v}_1^1$). (b) By consuming the EPR shared between the two super-nodes, a remote CZ operation is performed. This results in the generation of the additional edge $(\dot{v}_1^1, \ddot{v}_1^1)$, represented by the purple wavy line. (c) By re-coloring the overall graph, it becomes evident that it is a two-colorable graph corresponding to the binary star S_{n_1,n_2}.

and with edge-set E in (8). The proof follows by acknowledging that (8) coincides with (5).

Appendix B
Proof of Proposition 2

Let us adopt Fig. 3 labeling and let us suppose that the state $|K_{n_i}\rangle$ to be obtained must interconnect the clients of the right-most QLAN, which implies $n_i = n_2$. Accordingly, the final state must be the complete graph state $|K_{n_2}\rangle$, which corresponds to the complete graph $K_{n_2} = (P_1, P_1^2)$, with P_1 defined in (9). The proof follows by performing: i) $(n_1 - 1)$-Pauli-z measurements on the qubits stored at the clients of QLAN-1, and ii) a Pauli-y measurement on the qubit stored at the super-node of the second QLAN. From (2), the action of $(n_1 - 1)$-Pauli-z measurements is equivalent to remove all the client vertices in P_2 in (10), which yields to the graph:

$$S_{n_1,n_2} - P_2 \setminus \{\ddot{v}_1^1\} = \big(P_1 \cup \{\ddot{v}_1^1\}, P_1 \times \{\ddot{v}_1^1\}\big) = S'_{n_2+1} \quad (11)$$

We observe that the graph S'_{n_2+1} in (11) corresponds to a star graph connecting the vertices in the set $P_1 \cup \{\ddot{v}_1^1\}$, with the super-node $\ddot{v}_1^1$ being the center of the star. Then, a Pauli-y measurement is performed on the qubit stored at super-node of second QLAN and associated to the vertex $\ddot{v}_1^1$. From (2), the action of this measurement is equivalent to the local complementation of the graph S'_{n_2+1} at vertex $\ddot{v}_1^1$, followed by the deletion of $\ddot{v}_1^1$ from the graph, i.e., $\tau_{\ddot{v}_1^1}\left(S'_{n_2+1}\right) - \ddot{v}_1^1$. Step-by-step, we first perform the local complementation $\tau_{\ddot{v}_1^1}\left(S'_{n_2+1}\right)$, which yields to the graph:

$$\tau_{\ddot{v}_1^1}\left(S'_{n_2+1}\right) = \big(P_1 \cup \{\ddot{v}_1^1\}, (P_1 \cup \{\ddot{v}_1^1\})^2\big). \quad (12)$$

This is equivalent to add to the edge-set of the star graph S'_{n_2+1} all the possible edges having both endpoints in the subset P_1. As a result, we obtain the complete graph connecting the set $P_1 \cup \{\ddot{v}_1^1\}$. We then proceed by removing the super-node $\ddot{v}_1^1$, and the proof follows.

Appendix C
Proof of Proposition 3

Similarly to the previous proof, we adopt Fig. 3 labeling and suppose that the state to be obtained is shared among all the clients of the right-most QLAN. Accordingly, the final complete graph state is $|K_{n_2}\rangle$ corresponding to complete graph $K_{n_2} = \big(\{\dot{v}_j^2\} \cup \ddot{P}_2, (\{\dot{v}_j^2\} \cup \ddot{P}_2)^2\big)$ with $\ddot{P}_2$ defined in (7). The proof follows by performing: i) $(n_1 - 2)$-Pauli-z measurements on the qubits stored at the clients of the left-most QLAN, with the exception of the client $\dot{v}_j^2$, ii) a Pauli-y measurement at the super-node of the first QLAN, and iii) a Pauli-y measurement at the super-node of the second QLAN. From (2), by performing $(n_1 - 2)$ Pauli-z measurements on the clients is equivalent to remove all the clients in $\dot{P}_2$ in (10), except for the client node $\dot{v}_j^2$. Thus the resulting graph is:

$$\begin{aligned} &S_{n_1,n_2} - \big(\dot{P}_2 \setminus \{\dot{v}_j^2\}\big) = \\ &= \big(\underbrace{P_1 \cup \{\dot{v}_j^2, \ddot{v}_1^1\}}_{V'}, \underbrace{\{\ddot{v}_1^1\} \times P_1 \cup \{(\dot{v}_j^2, \dot{v}_1^1)\}}_{E'}\big) \stackrel{\triangle}{=} G' \end{aligned} \quad (13)$$

Then, a Pauli-y measurement is performed on at vertex $\dot{v}_1^1$ at super-node of the first QLAN. This yields to the graph:

$$\tau_{\dot{v}_1^1}(G') - \dot{v}_1^1 = \big(\ddot{P}_2 \cup \{\ddot{v}_1^1, \dot{v}_j^2\}, \{\ddot{v}_1^1\} \times (\ddot{P}_2 \cup \{\dot{v}_j^2\})\big) = S''_{n_2+1}. \quad (14)$$

(14) corresponds to a star graph connecting the set $\ddot{P}_2 \cup \{\ddot{v}_1^1, \dot{v}_j^2\}$ with super-node $\ddot{v}_1^1$ being the center of the star. Then, a Pauli-y measurement is performed on the qubit $\ddot{v}_1^1$ stored at super-node of second QLAN. From (2), this is equivalent to perform the following sequence of graph operations $\tau_{\ddot{v}_1^1}\left(S''_{n_2+1}\right) - \ddot{v}_1^1$. Step-by-step, the local complementation yields to the graph:

$$\tau_{\ddot{v}_1^1}\left(S''_{n_2+1}\right) = \big(\ddot{P}_2 \cup \{\dot{v}_j^2, \ddot{v}_1^1\}, (\ddot{P}_2 \cup \{\dot{v}_j^2, \ddot{v}_1^1\})^2\big) \quad (15)$$

This equals to adding to the edge-set of the star graph S''_{n_2+1} all the possible edges among the subset $\ddot{P}_2$ with node $\{\dot{v}_j^2\}$.

As a result, we obtain the complete graph connecting the set $\ddot{P}_2 \cup \{\ddot{v}_1^1, \dot{v}_j^2\}$, namely, the client nodes of the second QLAN, one client node $\{\dot{v}_j^2\}$ and the super-node of the first QLAN. Then, by removing the super-node $\ddot{v}_1^1$, the proof follows.